Introduction to Kinetic Theory

Electrostatic Waves

- Each species (electrons and ions) has a distribution function:

$$f_s = \underbrace{f_{s0}}_{\text{Equilibrium}} + \underbrace{f_{s1}}_{\text{perturbation}}$$

- For each species:

$$\int d\vec{v}\, f_{s0}(\vec{v}) = n_0$$

- Assuming waves propagating in the x-direction leads to:

$$\frac{\partial f_s}{\partial t} + v_x \frac{\partial f_s}{\partial x} + \frac{q_s}{m_s} E \frac{\partial f_s}{\partial v_x} = 0 \qquad (1)$$

- f_{s0} is a zeroth order quantity; f_{s1} and E are first order quantities.

  The zeroth order terms of (1) is:

$$\frac{\partial f_{s0}}{\partial t} + v_x \frac{\partial f_{s0}}{\partial x} = 0 \Rightarrow f_{s0}(\vec{v}) = f_{s0}$$

recall: this is now an independent variable (not zero or first order!)

- First order terms of (1):

$$\frac{\partial}{\partial t} f_{s1} + v_x \frac{\partial f_{s1}}{\partial x} + \frac{q_s E}{m_s} \frac{\partial f_{s0}}{\partial v_x} = 0$$

- We now look for plane wave solutions:

$$- i\omega f_{s1} + i v_x k f_{s1} + \frac{q_s E}{m_s} \frac{\partial f_{s0}}{\partial v_x} = 0 \iff$$

$$\iff \quad f_{s1} = \frac{1}{i(\omega - k v_x)} \frac{q_s E}{m_s} \frac{\partial f_{s0}}{\partial v_x}$$

- Couple this with Poisson:

$$i k E = \frac{e}{\varepsilon_0} (n_i - n_e) =$$

$$= \frac{e}{\varepsilon_0} \int d\vec{v} \, (f_i - f_e) = \frac{e}{\varepsilon_0} \int d\vec{v} \, (f_{i1} - f_{e1})$$

This is a plus!

$$= - \frac{e^2 i}{\varepsilon_0} E \int d\vec{v} \left(\frac{\partial f_{i0}}{\partial v_x} \frac{1}{m_i} + \frac{\partial f_{e0}}{\partial v_x} \frac{1}{m_e} \right) \frac{1}{\omega - k v_x} \iff$$

- Which we can re-write as:

This is a plus!

$$1 + \frac{e^2}{\varepsilon_0 k^2} \int\!\!\int\!\!\int dv_y \, dv_x \underbrace{\left(\frac{\partial f_{i0}}{\partial v_x} \frac{1}{m_i} + \frac{\partial f_{e0}}{\partial v_x} \frac{1}{m_e} \right)}_{= \frac{\partial}{\partial v_x} g} \frac{1}{\omega/k - v_x} = 0 \qquad (2)$$

This is a plus!

$$g = \int\!\!\int \left(f_{i0} \cdot \frac{1}{m_i} + f_{e0} \cdot \frac{1}{m_e} \right) dv_y \, dv_z$$

$$= \frac{1}{m_e} \int\!\!\int \left(f_{i0} \underbrace{\frac{m_e}{m_i}}_{m_e/m_i \text{ reduction for ion component}} + f_{e0} \right) dv_y \, dv_z$$

- Assuming Maxwellian electrons and ions:

$$f_{s0} = \frac{n_0}{(2\pi)^{3/2} v_s^3} \exp\left[- \frac{(v_x^2 + v_y^2 + v_z^2)}{2 v_s^2} \right]$$

$\hookrightarrow v_s^2 = K_b T_s / m_s$

- Integrations then give:

  - $$\int_{-\infty}^{+\infty} f_{s0} \, dv_y \propto \int \exp\left[- \frac{v_x^2 + v_y^2 + v_z^2}{2 v_s^2} \right] dv_y = v_s \sqrt{\pi \cdot 2}$$

  - $$\int_{-\infty}^{+\infty} f_{s0} \, dv_z \propto v_s \sqrt{2\pi}$$

This is a plus!

Thus:

$$\frac{1}{m_e} \int \Big(\underbrace{f_{i0} \frac{m_e}{m_i}} + \underbrace{f_{e0}} \Big) dv_y \, dv_z =$$

$$\frac{m_e}{m_i} \cdot v_i^2 \, \pi \cdot 2 \, \frac{n_0}{(2\pi)^{3/2} v_i^3} \cdot \exp\left(- \frac{v_x^2}{2 v_i^2} \right) \qquad v_e^2 \, 2\pi \, \frac{n_0}{(2\pi)^{3/2} v_e^3} \exp\left(- \frac{v_x^2}{2 v_e^2} \right) =$$

$$= \frac{m_e}{m_i} \frac{n_0}{\sqrt{2\pi}} \frac{1}{v_i} \exp\left(- v_x^2 / 2 v_i^2 \right) \qquad = \frac{n_0}{v_e \sqrt{2\pi}} \exp\left(- \frac{v_x^2}{2 v_e^2} \right)$$

This means that:

$$g = \left[\frac{m_e}{m_i} \times \frac{n_0}{\sqrt{2\pi}} \times \frac{1}{v_i} \exp\left(- \frac{v_x^2}{2 v_i^2} \right) + \frac{n_0}{v_e \sqrt{2\pi}} \exp\left(- \frac{v_x^2}{2 v_e^2} \right) \right] \frac{1}{m_e}$$

- Assume high frequency waves (ions) do not move. as $m_i \to \infty$. and restrict analysis for $\omega/k \gg v_x$ where $g(v_x)$ is appreciable so that $\partial_{v_x} g(v_x) = 0$ at $v_x = \omega/k$.

- Integrate by parts Eq. (2):

This is a plus!

$$1 + \frac{e^2 n_0}{\varepsilon_0 k^2 m_e} \frac{1}{v_e\sqrt{2\pi}} \int \frac{\overbrace{\frac{\partial}{\partial v_x} \exp\left(-\frac{v_x^2}{2v_e^2}\right)}^{h}}{\omega/k - v_x} dv_x = 0 \iff$$

this is a plus!

$$\iff \quad 1 + \frac{\omega_p^2}{k^2} \frac{1}{v_e\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\partial h/\partial v_x}{\omega/k - v_x} dv_x = 0$$

$$\begin{cases} \dfrac{\partial h}{\partial v_x} = F' \Rightarrow F = h\Big]_{-\infty}^{+\infty} = 0 \\[2ex] \dfrac{1}{\omega/k - v_x} = G \Rightarrow G' = -\dfrac{1}{(\omega/k - v_x)^2} \end{cases}$$

Thus, we obtain:

$$1 - \frac{\omega_p^2}{k^2} \frac{1}{v_e\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\exp(-v_x^2/2v_e^2)}{(\omega/k - v_x)^2} dv_x = 0 \iff$$

$$\iff \quad 1 - \frac{\omega_p^2}{k^2} \frac{1}{v_e\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{v_x^2}{2v_e^2}\right) \frac{k^2}{\omega^2} \frac{1}{(1 - v_x k/\omega)^2} = 0 \iff$$

$$\iff \quad 1 - \frac{\omega_p^2}{\omega^2} \frac{1}{\sqrt{2\pi}\, v_e} \int^{+\infty} \exp\left(-\frac{v_x^2}{2v_e^2}\right)\left[1 + \frac{2 v_x k}{\omega} + \frac{3 v_x^2 k^2}{\omega^2} + \dots\right] = 0 \quad (3)$$

The relevant integrations are thus:

- $$\int_{-\infty}^{+\infty} \exp\left(-\frac{v_x^2}{2v_e^2}\right) dv_x = \sqrt{2\pi}\, v_e$$

- $$\int_{-\infty}^{+\infty} \exp\left(-\frac{v_x^2}{2v_e^2}\right) v_x \, dv_x = 0$$

- $$\int_{-\infty}^{+\infty} \exp\left(-\frac{v_x^2}{2v_e^2}\right) v_x^2 \, dv_x = \sqrt{2\pi} \cdot v_e^3$$

Hence, Eq. (3) becomes:

$$1 - \frac{\omega_p^2}{\omega^2}\left[1 + \frac{3K^2 v_e^2}{\omega^2}\right] = 0 \iff$$

$$\omega^4 - \omega_p^2 \omega^2 - 3K^2 v_e^2 \omega_p^2 = 0 \iff$$

$$\iff \omega^2 = \frac{\omega_p^2 \pm \sqrt{\omega_p^4 + 4 \cdot 3K^2 v_e^2 \omega_p^2}}{2} =$$

$$= \frac{\omega_p^2 \pm \omega_p^2 \sqrt{1 + 12 K^2 v_e^2 / \omega_p^2}}{2} \simeq$$

$$\simeq \frac{\omega_p^2 \pm \omega_p^2 \left(1 + 6K^2 v_e^2/\omega_p^2\right)}{2} \overset{\sim}{\iff}$$

$$\begin{cases} \omega^2 = \omega_p^2 + \dfrac{3k^2 v_e^2}{\omega_p^2} & \leftarrow \text{Langmuir wave dispersion relation} \\[2ex] \underbrace{\omega^2 = -3 v_e^2 k^2 \quad \leftarrow \text{non-physical?}}_{\text{neglect this solution}} \end{cases}$$