Date: 6th January 2020
Duration: 1h30 (Test I/II) / 3h00 (Exam)
Jorge Vieira, Hugo Terças

Plasma Physics and Technology

MEFT 2019/20 – **EXAM I**

Carefully justify your answers and present in detail all the calculations you make.
Test I: Problems 1,2,3 **Test II:** Problems 4,5,6. **Exam:** ALL Problems

1. [2 val] Consider a homogeneous, quasineutral plasma where the ions and the electrons follow a power law (Tsallis) distribution, $n_{e,i} = n_0 \left[1 \pm (\alpha - 1)\frac{e\phi}{k_B T_{e,i}}\right]^{(\alpha+1)/2(\alpha-1)}$. This distribution seems to accurately describe the equilibrium of some regions of Saturn's magnetosphere.

(a) [1.0 val] Show that the Debye length can be given as

$$\frac{1}{\lambda_D^2} = \frac{1+\alpha}{2}\left(\frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2}\right).$$

(b) [1.0 val] Discuss the screening of a test charge in the cases $\alpha \to 1$ and $\alpha \to -1$. [1]

2. [4 val] Consider a homogeneous plasma composed of electrons (density n_-, charge $-e$, mass m_e) and positrons (density n_+, charge e, mass m_e) only. Electron-positron plasmas exist in astrophysical conditions, such as the magnetic poles of pulsars. We are interested in obtaining one-dimensional, electrostatic oscillations supported in such a system.

(a) [1.5 val] Starting from the fluid equations for both species, show that the linearized problem ($n_\alpha \simeq n_{0\alpha} + n_{1\alpha}$, with $\alpha = \pm$) leads to

$$(\omega^2 - \gamma_\alpha v_\alpha^2 k^2) n_{\alpha 1} = i\alpha k \frac{e n_{0\alpha}}{m_e} E_1,$$

where γ_α is a constant.

(b) [1.5 val] Assume quasineutrality ($n_{0+} = n_{-0} \equiv n_0$) and consider that both electrons and positrons are in thermal equilibrium, $T_+ = T_- \equiv T_e$. Define the quantity $\delta_1 = n_{+1} - n_{-1}$ and show that it yields a mode of frequency

$$\omega^2 = 2\omega_{pe}^2 + 3v_e^2 k^2.$$

Discuss the result physically, namely the appearance of the numerical factor.

[1] In fact, some quantum mechanical effects take place when $\alpha \to -1$.

(c) [1.0 val] Repeat the procedure on the previous point by defining the quantity $N_1 = n_{+1} + n_{-1}$ to show that electron-positron plasmas support an acoustic mode of dispersion

$$\omega = \sqrt{3} v_e k.$$

Explain this result physically and establish the differences with respect to the ion acoustic waves in electron-ion plasmas.

3. [4 val] Consider electromagnetic waves propagating along $\mathbf{k} = k\mathbf{e}_z$ in a plasma composed of electrons and immobile ions. The plasma is magnetised with an external magnetic field along the electromagnetic wave propagation direction, $\mathbf{B}_0 = B_0\mathbf{e}_z$. The index of refraction, n, for these waves is given by:

$$n^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1 \pm \Omega_e/\omega}, \tag{1}$$

where ω is the frequency of a given mode, $\omega_p = \sqrt{e^2 n_0/(m_e \epsilon_0)}$ the electron plasma frequency in a plasma with uniform background density n_0, $\Omega_e = -eB_0/m_e$ (< 0) is the electron cyclotron frequency, e the elementary charge and m_e the electron mass. These modes are circularly polarised. The $\pm$ sign respectively correspond to the R$-$ and L$-$waves, which are right and left handed circularly polarised. Electromagnetic waves in these conditions are traditionally characterised by several important properties, such as cut-offs and resonances. The goal of this problem is to recover some of these properties from the expression for the index of refraction given by Eq. (1).

(a) [1 val] Define cutoff mathematically and physically. Use Eq. (1) to determine the frequencies for which there are cutoffs.

(b) [1 val] Define resonance mathematically and physically. Use Eq. (1) to determine the resonant frequency. Justify the presence of this resonance.

(c) [2 val] Sketch $n^2(\omega)$ clearly indicating the resonant and cutoff frequencies for the different branches (R$-$ and L$-$waves). Assume the cutoff frequencies for the R$-$ and L$-$ waves, ω_R and ω_L respectively, follow the following relation $\omega_L < \omega_p < \omega_R$.

4. [4 val] Consider a weakly ionized plasma composed of electrons, positrons and neutrals. In order to describe the diffusive processes, we assume that both electrons and positron collide with the neutrals at a constant rate ν. In what follows, we neglect the motion of the neutrals.

(a) [1.5 val] Write down the condition for ambipolar diffusion to take place in a plasma in terms of the positron and electron fluxes, $\vec{\Gamma}_+$ and $\vec{\Gamma}_-$. In the conditions of ambipolar diffusion, show that the particle flux and the density gradient for each species is given as

$$\vec{\Gamma} = -D_a \vec{\nabla} n,$$

where $D_a = D_- = D_+$ and n is the density of either species (electrons or positrons). Justify.

(b) [1.0 val] Define the total density $N = n_+ + n_-$. Show that it follows the following diffusion equation

$$\frac{\partial N}{\partial t} - 2D_a \nabla^2 N = 0. \tag{2}$$

(c) [0.5 val] Apply a Fourier transformation to the previous result to obtain the following diffusive mode

$$\omega = -2iD_a k^2.$$

Interpret this result physically and explain its relation with diffusive processes.

(d) [1.0 val] Consider that the plasma is contained between two infinite, parallel planes located at $x = 0$ and $x = L$. We are interested in investigating the structure of the modes. As such, we look for solutions of the form $N(x,t) = \sum_\ell A_\ell e^{-i\omega_\ell t} f_\ell(x)$. Solve the differential equation (2) to show that $f_\ell(x) = \sin(k_\ell x)$, with $k_\ell = \ell\pi/L$, and make use of the result in point c) to determine $|\omega_\ell|$, i.e. the rate at which the modes decay in time. Sketch the first two modes ($\ell = 1, 2$) graphically and explain the results physically.

5. [3 val] There are several descriptions for the plasma, such as the fluid and kinetic descriptions. The goal of this exercise is to recall some of the key features associated with each of these descriptions.

(a) [1.0 val] In both the kinetic and fluid descriptions, it is often useful to linearize the Vlasov equation or the fluid equations. The first order quantities in the linearized equations are essentially the same, but there are key differences. Distinguish between the notion of velocity in the fluid and kinetic description of the plasma. Does it makes sense to make perturbations to the velocity in kinetic theory? Justify your answers.

(b) [1.0 val] The MHD description of the plasma is valid in strongly collisional regimes. One particularly important quantity in these scenarios is the plasma electrical conductivity, σ, and resistivity, η. Consider Newton's law for an electron in an external electric field $\vec{E}$. Find an estimate for σ and η as a function of the electron-ion collision time, τ_{ei} considering that the collisional term is given by $(dp/dt)_{\text{coll}} = -m_e\langle v_e\rangle/\tau_{ei}$, where $\langle v_e\rangle$ is the average electron velocity, and m_e the electron mass. Justify your derivation. [Hint: consider a steady-state situation in your calculations.]

(c) [1.0 val] Consider the plasma dielectric constant for electrostatic ion and electron waves in 1D, which is given by:

$$\epsilon = 1 - \frac{\omega_{pi}^2}{k^2}\int \mathrm{d}v_x \frac{f_{ix}}{(v_x - \omega/k)^2} - \frac{\omega_{pe}^2}{k^2}\int \mathrm{d}v_x \frac{f_{ex}}{(v_x - \omega/k)^2}, \tag{3}$$

where $f_{(i,e)x} = 1/(v_{i,e}\sqrt{2\pi})\exp\left[-v^2/(2v_{i,e}^2)\right]$, where $v_{i,e}$ is the thermal speed for ions/electrons. Solving the corresponding dispersion relation is generally complicated because of the presence of poles that need to be treated with care. Explain, in a purely qualitative but detailed way, the contribution of these poles to the amplitude of the plasma waves.

6. [3 val] In class, we have used fluid theory to determine the dispersion relation of electrostatic waves in 1D. The goal of this problem is to compute a general dispersion relation for these 1D electrostatic waves from first principles using kinetic theory and to compare with the fluid theory results.

(a) [1.0 val] Consider Eq. (3). Neglect the contribution from the poles and assume that $k_B T_i/m_i \ll \omega^2/k^2 \ll k_B T_e/m_e$. Without evaluating the integrals from Eq. (3), re-write the dielectric constant of the plasma in these limits by performing a suitable Tailor expansion of the arguments of each one of the integrals in ϵ. [Hint: Notice that $(1+x)^n \simeq 1 + nx + x^2 n(n-1)/2 + \mathcal{O}(x^3)$ and retain terms up to second order in x for the ions and zeroth order in x for the electrons ($x \ll 1$ is a small parameter)].

(b) [1.0 val] By evaluating the integrals from (a), show that the dielectric constant is given by:

$$\epsilon = 1 - \frac{\omega_{pi^2}}{\omega^2}\left(1 + \frac{3k^2}{\omega^2}\frac{k_B T_i}{m_i}\right) + \frac{1}{k^2\lambda_D^2}, \tag{4}$$

where λ_D is the Debye length. The following results are useful: $\int_{-\infty}^{\infty} \exp(-v^2/(2v_s^2) = \sqrt{2\pi} v_s$, $\int_{-\infty}^{\infty} v^2 \exp(-v^2/2v_s^2) = \sqrt{2\pi} v_s^3$ and $\int_{-\infty}^{\infty} \exp(-v^2/2v_s^2)/v^2 = -\sqrt{2\pi}/v_s$

(c) [1.0 val] Starting from Eq. (4), write down the dispersion relation in the limits (i) $k\lambda_D \gg 1$ and $T_i \to 0$ and (ii) $k\lambda_D \ll 1$. Based on your previous knowledge, what waves are excited in these limits? Comment the results based on the fluid description of the plasma.

- Constants and mathematical relations:

$$m_e = 9.1 \times 10^{-31} \text{ kg}; \quad e = 1.6 \times 10^{-19} \text{ C}; \quad \epsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$$

- Drifts and fundamental effects

$$\lambda_e = \sqrt{\frac{\varepsilon_0 k T_e}{n_e e^2}} \qquad \omega_{pe} = \sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}}$$

$$\omega_{ce} = \frac{eB}{m_e} \qquad r_L = \frac{v_\perp}{\omega_{ce}}$$

$$\text{ExB drift} \qquad \vec{v}_d = \frac{\vec{E} \times \vec{B}}{B^2}$$

$$\text{Grad B drift} \qquad \vec{v}_d = \frac{m v_\perp^2}{2qB}\frac{\vec{B} \times \vec{\nabla} B}{B^2}$$

$$\text{Curvature drift} \qquad \vec{v}_d = \frac{m v_\parallel^2}{qB^2}\frac{\vec{u}_r \times \vec{B}}{R_c}$$

$$\text{Fields in vacuum} \qquad \vec{v}_d = \left(m v_\parallel^2 + \frac{1}{2}v_\perp^2\right)\frac{1}{qB^2}\frac{\vec{u}_r \times \vec{B}}{R_c}$$

$$\text{Polarization drift} \qquad \vec{v}_d = \frac{m}{qB^2}\frac{d}{dt}\vec{E}_\perp$$

$$\text{External force drift} \qquad \vec{v}_d = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2}$$

- Fluid equations

$$\frac{\partial}{\partial t} n_s + \vec{\nabla} \cdot (n_s \vec{v}_s) = 0$$

$$n_s m_s \left[\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s\right] = q_s n_s \left[\vec{E} + \vec{v}_s \times B\right] - \vec{\nabla} P_s - \nu_s n_s m_s (\vec{v}_s - \vec{v}_0)$$

$$D_\alpha = \frac{k_B T_\alpha}{m_\alpha \nu}$$

- Maxwell's equations

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \qquad & \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \qquad & \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}$$