Single Fluid MHD equations

- Equations to describe the plasma as a single fluid. Define mass velocity:

$$\vec{V} = \frac{n_i m_i \vec{v}_i + n_e m_e \vec{v}_e}{n_i m_i + n_e m_e} \qquad (1)$$

- The current as:

$$\vec{j} = e\left(z n_i \vec{v}_i - n_e \vec{v}_e \right) \qquad (2)$$

- The density as:

$$\rho = n_i m_i + n_e m_e \qquad (3)$$

- We also need continuity:

$$\frac{\partial n_e}{\partial t} = - \vec{\nabla} \cdot (n_e \vec{v}_e) \qquad (4)$$

$$\frac{\partial n_i}{\partial t} = - \vec{\nabla} \cdot (n_i \vec{v}_i) \qquad (5)$$

- Multiply (4) by m_e and (5) by m_i leads to:

$$\begin{cases} m_e \dfrac{\partial n_e}{\partial t} = -\vec{\nabla}\cdot(m_e n_e \vec{v}_e) \\ \\ m_i \dfrac{\partial n_i}{\partial t} = -\vec{\nabla}\cdot(m_i n_i \vec{v}_i) \end{cases} \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial t}\underbrace{(m_e n_e + m_i n_i)}_{\rho} = -\vec{\nabla}\cdot(\underbrace{m_e n_e \vec{v}_e + m_i n_i \vec{v}_i}_{\rho\vec{v}}) \Leftrightarrow$$

$$\Leftrightarrow \boxed{\frac{\partial}{\partial t}\rho = -\vec{\nabla}\cdot(\rho\vec{v})}$$

- We also define the charge density as:

$$\sigma = e(n_i z - n_e)$$

- Multiply (4) by $-e$ and (5) by ze and add leads to:

$$\begin{cases} -\dfrac{\partial}{\partial t} e n_e = +\vec{\nabla}\cdot(e n_e v_e) \\ \\ \dfrac{\partial z e n_i}{\partial t} = -\vec{\nabla}\cdot(z e n_i v_i) \end{cases} \Rightarrow$$

$$\frac{\partial}{\partial t}\underbrace{\left(z e n_i - e n_e \right)}_{\sigma} = -\vec{\nabla}\cdot\left(\underbrace{z e n_i \vec{v}_i - e n_e \vec{v}_e}_{\vec{j}} \right) \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{\partial \sigma}{\partial t} = -\vec{\nabla}\cdot\vec{j} \quad \Leftrightarrow \quad \boxed{\frac{\partial \sigma}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}$$

- Approximations:

  - $$1 \gg \underbrace{\frac{z m_e}{m_i}}$$

$$\frac{\frac{q_i}{m_i}}{\frac{q_e}{m_e}} = \frac{\frac{ze}{m_i}}{\frac{e}{m_e}} = \frac{ze}{m_i}\frac{m_e}{e} = \frac{z m_e}{m_i}$$

  - $n_e = z n_i \equiv n$ (quasi-neutral plasma)

(this does not mean that σ is zero! $\sigma \ll n_e e$)

- Equations for $\vec{V}$ and j become:

$$\begin{cases} \vec{V} \simeq \vec{v}_i + \dfrac{m_e z \vec{v}_e}{m_i} & (6) \\ & \text{or} \\ j \simeq e n_e (\vec{v}_i - \vec{v}_e) & (7) \end{cases}$$

- $\dfrac{-\vec{j} + e n_e \vec{v}_i}{e n_e} = \vec{v}_e \Leftrightarrow \vec{v}_e = \vec{v}_i - \dfrac{\vec{j}}{e n_e}$ (from (7))

- $\vec{V} - \dfrac{m_e Z}{m_i}\vec{v}_e = \vec{v}_i \Leftrightarrow \vec{V} - \dfrac{m_e Z}{m_i}\left(\vec{v}_i - \dfrac{\vec{j}}{e n_e}\right) = \vec{v}_i \Leftrightarrow$

$$\Leftrightarrow \boxed{\vec{v}_i \left(1 + \underbrace{\cancel{\frac{m_e Z}{m_i}}}_{\ll 1}\right) = \vec{V} + \frac{m_e Z}{m_i n_e e}\vec{j}} \quad (8)$$

- and, using (8):

$$\boxed{\vec{v}_e = \vec{V} + \left(\underbrace{\cancel{\frac{m_e Z}{m_i n_e e}}}_{\ll \frac{1}{e n_e}} - \frac{1}{e n_e}\right)\vec{j}}$$

- Equation of motion neglecting $\vec{v}\cdot\nabla\vec{v}$ (linear equations in $\vec{v}$) and also negleting viscosity tensor (assume stress tensor leading to isotropic pressure term)

$$m_i n_i \frac{\partial v_i}{\partial t} = Z e n_i(\vec{E} + \vec{v}_i \times \vec{B}) - \vec{\nabla} P_i + m_i n_i \vec{g} + \left.\frac{\partial p_{i-e}}{\partial t}\right)_{coll} \quad (9)$$

$$m_e n_e \frac{\partial v_e}{\partial t} = -e n_e(\vec{E} + \vec{v}_e \times \vec{B}) - \vec{\nabla} P_e + m_e n_e \vec{g} + \left.\frac{\partial p_{e-i}}{\partial t}\right)_{coll} \quad (10)$$

Add Eq. (9) and (10):

$$m_i n_i \frac{\partial \vec{v}_i}{\partial t} + m_e n_e \frac{\partial \vec{v}_e}{\partial t} = \vec{E}\,\underbrace{\left(z e \cancel{n_i} - \cancel{n_e}\right)}_{=0} + \left(z e n_i \vec{v}_i - e n_e \vec{v}_e\right) \times \vec{B} +$$

$$+ \left(m_i n_i + m_e n_e\right)\vec{g} - \vec{\nabla} p$$

- E-field cancels out
- $\left.\frac{\partial P_{e,i}}{\partial t}\right)_{coll} = - \left.\frac{\partial P_{i,e}}{\partial t}\right)_{coll}$
- $p = p_i + p_e$

Using Eqs. (5) and (2) leads to:

$$m_i n_i \frac{\partial}{\partial t}\left[\vec{v} + \frac{m_e z}{m_i n_e e}\vec{j}\right] + m_e n_e \frac{\partial}{\partial t}\left[\vec{v} - \frac{\vec{j}}{n_e e}\right] = \underbrace{e n\left(\vec{v}_i - \vec{v}_e\right)}_{\vec{j}} \times \vec{B} +$$

$$+ m \rho \vec{g} - \vec{\nabla} p \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \underbrace{\left(m_i n_i + m_e n_e\right)}_{=\rho} \frac{\partial \vec{v}}{\partial t} + \underbrace{n_i \frac{\partial}{\partial t}\left(\frac{m_e z}{n_e e}\vec{j}\right) - n_i \frac{\partial}{\partial t}\left(\frac{\vec{j}\, m_e z}{n_e e}\right)}_{=0} = \vec{j} \times \vec{B} +$$

$$+ m \rho \vec{g} - \vec{\nabla} p \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \boxed{\rho \frac{\partial \vec{v}}{\partial t} = \vec{j} \times \vec{B} + m \rho \vec{g} - \vec{\nabla} p}$$

Multiply (9) by $e\mathfrak{z}/m_i$ and (10) by $-e/m_e$ and add:

$$\underbrace{e \mathfrak{z} m_i \frac{\partial \vec{v}_i}{\partial t} - e m_e \frac{\partial \vec{v}_e}{\partial t}}_{\text{first term}} = \underbrace{\left(\frac{e^2 \mathfrak{z}^2}{m_i} m_i + \frac{e^2 m_e}{m_e} \right)}_{\text{second term}} \vec{E} +$$

$$+ \underbrace{\left(\frac{e^2 \mathfrak{z}^2}{m_i} m_i \vec{v}_i + \frac{e^2 m_e}{m_e} \vec{v}_e \right)}_{\text{third term}} \times \vec{B} -$$

$$- \underbrace{\frac{e\mathfrak{z}}{m_i} \vec{\nabla} p_i + \frac{e}{m_e} \vec{\nabla} p_e}_{\text{fourth term}} + \underbrace{\left(e \mathfrak{z} m_i - m_e e \right) \vec{g}}_{\text{fifth term}} +$$

$$+ \underbrace{\frac{e\mathfrak{z}}{m_i} \left.\frac{\partial p_{ie}}{\partial t}\right)_{coll} - \frac{e}{m_e} \left.\frac{\partial p_{e\text{-}i}}{\partial t}\right)_{coll}}_{\text{sixth term}}$$

From Eq. (2):

$$j = e \left(\mathfrak{z} m_i \vec{v}_i - m_e \vec{v}_e \right) =$$

$$\simeq e \mathfrak{z} m_i \left(\vec{v}_i - \vec{v}_e \right) \Rightarrow \vec{j} / e\mathfrak{z} m_i \simeq \vec{v}_i - \vec{v}_e$$

<u>Hence the first term is:</u>

$$e \mathfrak{z} m_i \frac{\partial \vec{v}_i}{\partial t} - e m_e \frac{\partial \vec{v}_e}{\partial t} \simeq e \mathfrak{z} m_i \frac{\partial \vec{v}_i - \vec{v}_e}{\partial t} \simeq m_i \frac{\partial (j/m_i)}{\partial t} = \boxed{n \frac{\partial}{\partial t} \left(\frac{\vec{j}}{n} \right)}$$

<u>The second term is:</u>

$$\left(\frac{e^2 \mathfrak{z}^2 m_i}{m_i} + \frac{e^2 m_e}{m_e} \right) \vec{E} \simeq e^2 \mathfrak{z} m_i \left(\frac{\mathfrak{z}}{m_i} + \frac{1}{m_e} \right) \vec{E} \simeq \frac{e^2 \mathfrak{z} m_i}{m_e} \vec{E} \simeq \boxed{\frac{e^2 n}{m_e} \vec{E}}$$

The third term is:

$$\left(\frac{e^2 z^2}{m_i} m_i \vec{v}_i + \frac{e^2 m_e}{m_e} \vec{v}_e\right) \times \vec{B} =$$

$$= \left[\frac{e^2 z^2}{m_i} m_i \left(\vec{V} + \frac{m_e z}{m_i m_e e} \vec{j}\right) + \frac{e^2 m_e}{m_e}\left(\vec{V} - \frac{\vec{j}}{e m_e}\right)\right] \times \vec{B} =$$

$$= \left(\frac{e^2 z^2 m_i}{m_i} + \frac{e^2 m_e}{m_e}\right) \vec{V} \times \vec{B}, + \left(\frac{e^2 z^2 n_i m_e z}{m_i^2 n_e e} - \frac{e^2 n_e}{m_e n_e e}\right) \vec{j} \times \vec{B} \simeq$$

$$\simeq e^2 z n_i \left(\frac{z}{m_i} + \frac{1}{m_e}\right) \vec{V} \times \vec{B} + m_e e \left(\frac{z^2}{m_i^2} - \frac{1}{m_e^2}\right) \vec{j} \times \vec{B} \simeq$$

$$\simeq \boxed{\frac{e^2 n}{m_e} \vec{V} \times \vec{B} - \frac{e}{m_e} \vec{j} \times \vec{B}}$$

The fourth term is:

$$- \frac{e z}{m_i} \vec{\nabla} p_i + \frac{e}{m_e} \vec{\nabla} p_e \simeq \boxed{\frac{e}{m_e} \vec{\nabla} p_e}$$

The fifth term is:

$$(e z m_i - m_e e) g \simeq \boxed{0}$$

The sixth term is:

$$\frac{e\,z}{m_i}\left.\frac{\partial p_{i\text{-}e}}{\partial t}\right)_{coll} - \frac{e}{m_e}\left.\frac{\partial p_{e\text{-}i}}{\partial t}\right)_{coll} = \frac{e\,z}{m_i}\alpha\left(\vec{v}_e - \vec{v}_i\right) + \frac{e}{m_e}\alpha\left(\vec{v}_e - \vec{v}_i\right)$$

$$\simeq \frac{e}{m_e}\alpha\left(\vec{v}_e - \vec{v}_i\right) \simeq \boxed{-\frac{\alpha\vec{j}}{m_e n}} \quad ; \quad \alpha = n_e m_e \nu_{e\text{-}i}$$

Divide all terms by $e^2 n/m_e$ and put together:

$$\frac{m_e}{e^2}\frac{\partial}{\partial t}\left(\frac{\vec{j}}{n}\right) = \vec{E} + \vec{V}\times\vec{B} - \frac{e}{m_e}\cdot\frac{m_e}{ne^2}\vec{j}\times\vec{B} + \frac{e}{m_e}\frac{m_e}{ne^2}\vec{\nabla}p_e$$

$$+ \frac{\alpha}{m_e n}\cdot\frac{m_e}{ne^2}\alpha\vec{j} =$$

$$= \vec{E} + \vec{V}\times\vec{B} - \frac{1}{ne}\vec{j}\times\vec{B} + \frac{1}{ne}\vec{\nabla}p_e - \underbrace{\frac{\alpha}{n^2e^2}}_{\eta\ \text{(resistivity)}}\vec{j}$$

$$\frac{\alpha}{n^2e^2} = \frac{n m_e \nu_{ei}}{n^2e^2} = \frac{m_e}{ne^2}\nu_{e\text{-}i} \equiv \eta$$

In a steady state:

$$\underbrace{\vec{E} + \vec{V}\times\vec{B} = \frac{1}{ne}\left(\overbrace{\vec{j}\times\vec{B}}^{\text{Hall current}} - \vec{\nabla}p_e\right) + \eta\vec{j}}_{\text{generalized Ohm's Law}}$$

It is often the case that:

$$\boxed{\vec{E} + \vec{V}\times\vec{B} = \eta\vec{j}}$$

The full set of MHD equations is thus:

$$\rho \frac{\partial \vec{v}}{\partial t} = \vec{j} \times \vec{B} + m \rho \vec{g} - \vec{\nabla} p \qquad (11)$$

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j} \qquad (12)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \qquad (13)$$

$$\frac{\partial \sigma}{\partial t} + \hat{\nabla} \cdot \hat{j} = 0 \qquad (14)$$

- Where are these equations important?

  - Astrophysics

  - Fusion

- Current research hot topic:

  Can we recover the physics of MHD using kinetic equations?