Plasma Physics and Technology
Electrostatic waves in magnetised plasmas

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Magnetised plasmas

Waves in plasmas: general categorisation

- Key directions: $\mathbf{k}$ and $\mathbf{B}$
- Key wave quantities: $\mathbf{E}_1$ and $\mathbf{B}_1$
- 6 types of waves:
  - Parallel $\hat{\mathbf{k}} \cdot \hat{\mathbf{B}}_0 = 1$
  - Perpendicular $\hat{\mathbf{k}} \cdot \hat{\mathbf{B}}_0 = 0$
  - Longitudinal $\hat{\mathbf{k}} \cdot \hat{\mathbf{E}}_1 = 1$
  - Transverse $\hat{\mathbf{k}} \cdot \hat{\mathbf{E}}_1 = 0$
  - Electrostatic $\mathbf{B}_1 = 0$
  - Electromagnetic $\mathbf{B}_1 \neq 0$

Relations between different wave categories exist

- Recall Faraday's law

$$\mathbf{k} \times \mathbf{E}_1 = \omega \mathbf{B}_1$$

- Longitudinal waves are electrostatic and vice versa:

$$\mathbf{k} \times \mathbf{E}_1 = 0$$

- Transverse waves are electromagnetic

$$\mathbf{k} \times \mathbf{E}_1 \neq 0$$

Nicholson, pp. 144-145

Upper hybrid waves

Relevant equations

- Electrostatic electron plasma waves propagating across external B
  - $\mathbf{k} \cdot \mathbf{B}_0 = 0$
  - $\hat{\mathbf{k}} \cdot \hat{\mathbf{E}}_1 = 1$

- Relevant equations

$$\frac{\partial n_e}{\partial t} + \nabla \cdot \left(n_e \mathbf{v}_e\right) = 0$$

$$m_e n_e \frac{\partial \mathbf{v}_e}{\partial t} + m_e n_e \mathbf{v}_e \cdot \nabla \mathbf{v}_e = -e n_e \mathbf{E}_1 - e n_e \mathbf{v}_e \times \mathbf{B}_0$$

$$\nabla \cdot \mathbf{E} = \frac{e(n_0 - n_e)}{\epsilon_0}$$

Geometry and strategy to solve equations

- Geometry
  - $\mathbf{E}_1 = E_1 \mathbf{e}_x$
  - $\mathbf{k} = k \mathbf{e}_x$
  - $\mathbf{B}_0 = B_0 \mathbf{e}_z$

- Strategy:
  - Combine Poisson+Continuity to get $v_x(E_1)$
  - Use force equation along y combined with $v_x(E_1)$ to get $v_y(E_1)$
  - Substitute $v_x(E_1)$ and $v_y(E_1)$ in force equation along y

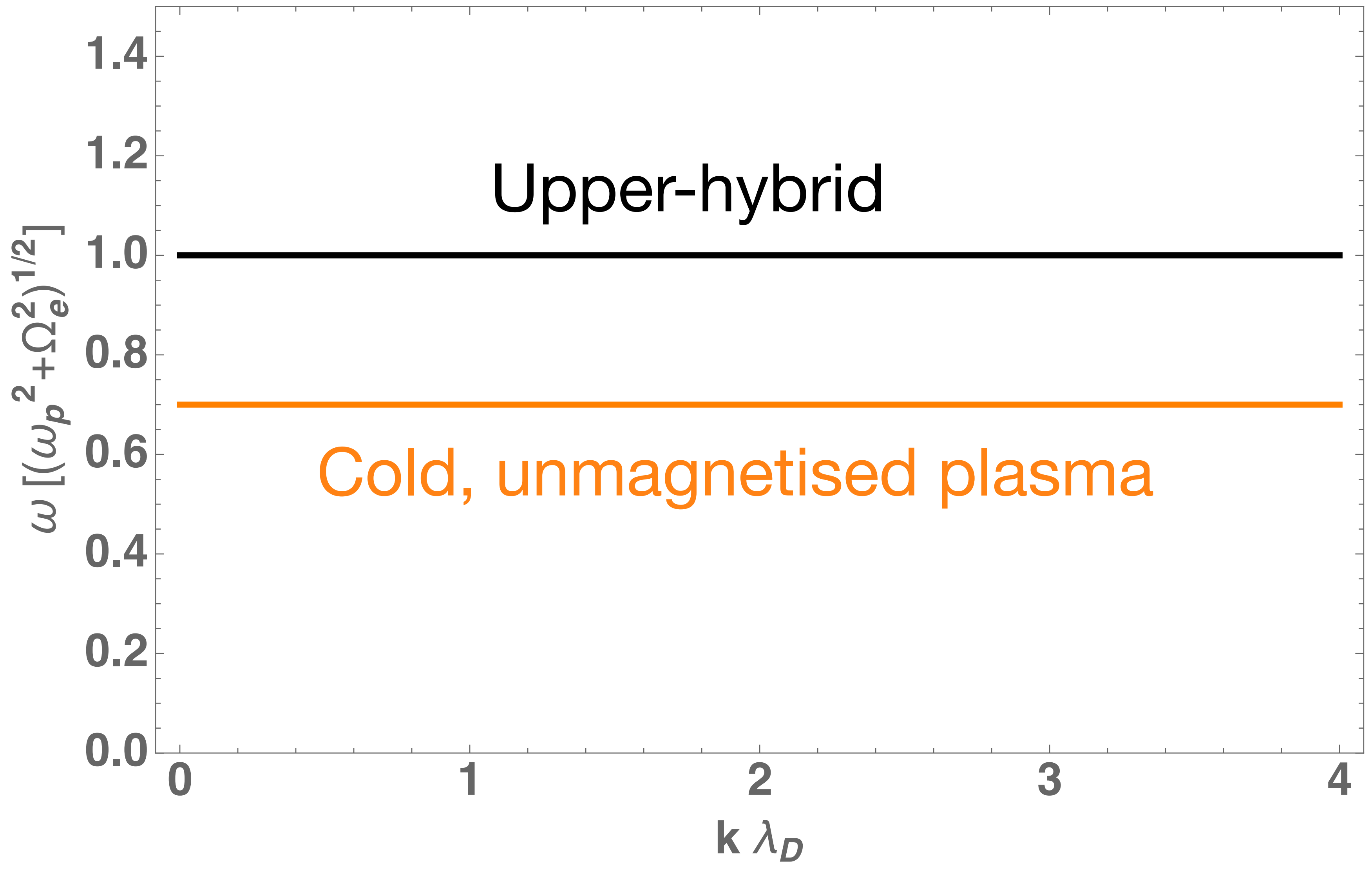
Nicholson, pp. 145-146

Upper hybrid waves - dispersion relation and properties

Dispersion relation

$$\omega^2 = \omega_p^2 + \Omega_e^2$$

$$\Omega_e^2 = \frac{-eB_0}{m_e}$$



Key properties

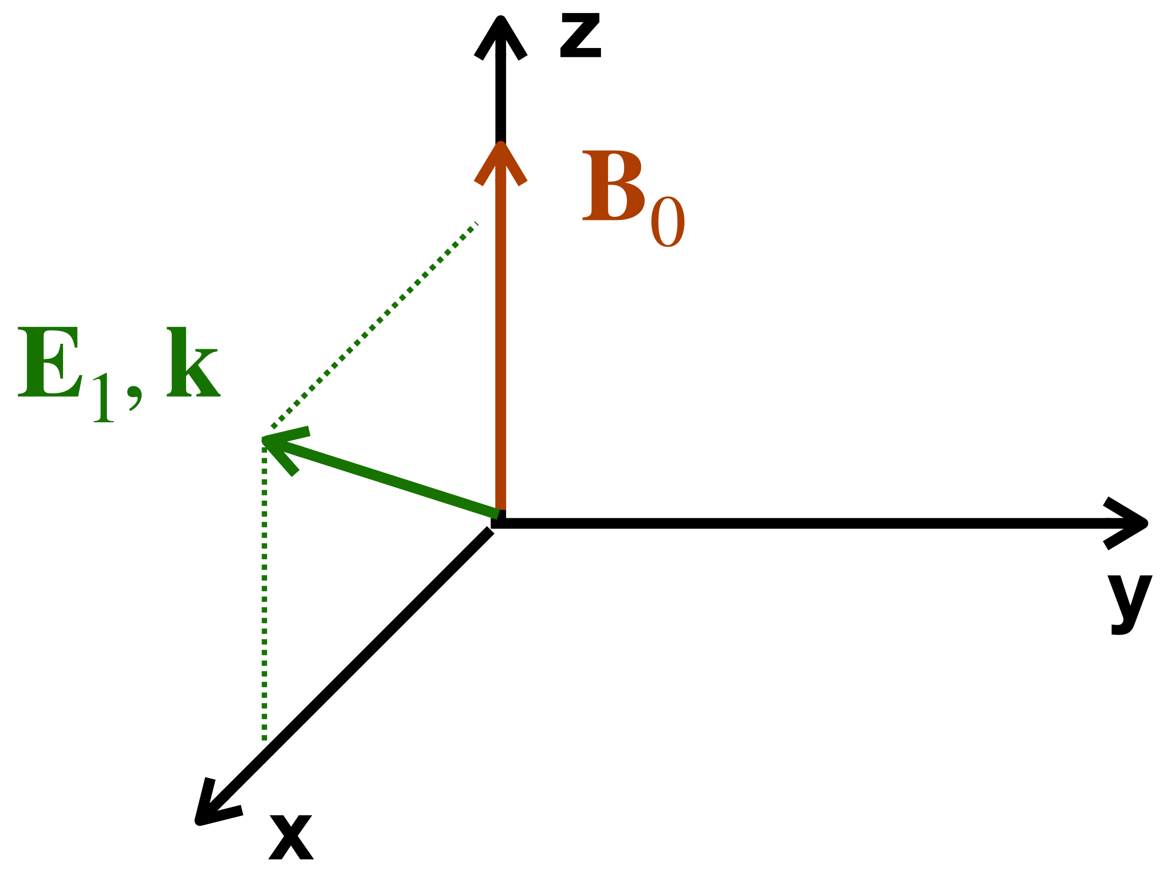
- Pure oscillations - $v_g = 0$ just as in the cold, unmagnetised plasma scenario
- When $n_e \to 0$ recover cyclotron motion
- When $B_0 \to 0$ recover longitudinal cold plasma oscillation
- **Elliptical** electron trajectories (major axis along x, minor axis along y) where $v_y = i\Omega_e/\omega v_x$

Nicholson, pp. 145-146

Electrostatic ion waves in magnetised plasmas

Geometry and assumptions

- Longitudinal waves ($\mathbf{k} \| \mathbf{E}_1$)
- Low frequency waves
- Use plasma approximation (quasi-neutrality) and discard Poisson's equation



Relevant equations

Electrons

$$m_e n_0 \frac{\partial \mathbf{v}_e}{\partial t} = -\gamma_e k_b T_e \nabla n_e - e n_0 \mathbf{E}_1 - e n_0 \mathbf{v}_e \times \mathbf{B}_0$$

$$\frac{\partial n_e}{\partial t} + n_0 \nabla \cdot \mathbf{v}_e = 0$$

Ions

$$m_i n_0 \frac{\partial \mathbf{v}_i}{\partial t} = -\gamma_i k_b T_i \nabla n_e - e n_0 \mathbf{E}_1 - e n_0 \mathbf{v}_i \times \mathbf{B}_0$$

$$\frac{\partial n_e}{\partial t} + n_0 \nabla \cdot \mathbf{v}_i = 0$$

Dawson, pp. 146-150

General dispersion relation

After a few algebraic calculations…

$$1 - \frac{\mathbf{k}^2 c_s^2}{\omega^2} + \frac{\Omega_i}{\omega}\left\{ -\left[\frac{\omega}{\Omega_i} - \frac{k_z^2}{k_x^2}\frac{\Omega_i}{\omega}\left(1 - \frac{\omega^2}{\Omega_i^2}\right)\right]^{-1} + \left[\frac{\omega}{\Omega_e} - \frac{k_z^2}{k_x^2}\frac{\Omega_e}{\omega}\left(1 - \frac{\omega^2}{\Omega_e^2}\right)\right]^{-1}\right\} = 0$$

$$c_s^2 = \frac{\gamma_e T_e + \gamma_i T_i}{m_i}$$

Dawson, pp. 146-150

Limiting cases: propagation along B

After a few algebraic calculations…

$$1-\frac{\mathbf{k}^2c_s^2}{\omega^2}+\frac{\Omega_i}{\omega}\left\{\cancel{-\left[\frac{\omega}{\Omega_i}-\frac{k_z^2}{k_x^2}\frac{\Omega_i}{\omega}\left(1-\frac{\omega^2}{\Omega_i^2}\right)\right]^{-1}}+\cancel{\left[\frac{\omega}{\Omega_e}-\frac{k_z^2}{k_x^2}\frac{\Omega_e}{\omega}\left(1-\frac{\omega^2}{\Omega_e^2}\right)\right]^{-1}}\right\}=0$$

$$c_s^2=\frac{\gamma_eT_e+\gamma_iT_i}{m_i}$$

Ion acoustic waves

Assume $\mathbf{k}\|\mathbf{e}_z$, $\omega\neq\pm\Omega_e$ and $\omega\neq\pm\Omega_i$

$$\omega=k_zc_S$$

Ion cyclotron wave

Assume $\mathbf{k}\|\mathbf{e}_z$, $\omega\neq\pm\Omega_e$ and $\omega=\Omega_i$

$$1-\frac{k_z^2c_s^2}{\omega^2}-\left[\frac{\omega}{\Omega_i}-\frac{k_z^2}{k_x^2}\frac{\Omega_i}{\omega}\left(1-\frac{\omega^2}{\Omega_i^2}\right)\right]^{-1}=0$$

Equation admits solution $\omega=\Omega_i$ by letting

$k_x\to 0$ and $\omega\to\Omega_i$

Dawson, pp. 146-150

Limiting cases: propagation perpendicular to B

After a few algebraic calculations…

$$1 - \frac{\mathbf{k}^2 c_s^2}{\omega^2} + \frac{\Omega_i}{\omega}\left\{ -\left[\frac{\omega}{\Omega_i} - \frac{k_z^2}{k_x^2}\frac{\Omega_i}{\omega}\cancel{\left(1 - \frac{\omega^2}{\Omega_i^2}\right)}\right]^{-1} + \left[\frac{\omega}{\Omega_e} - \frac{k_z^2}{k_x^2}\frac{\Omega_e}{\omega}\cancel{\left(1 - \frac{\omega^2}{\Omega_e^2}\right)}\right]^{-1}\right\} = 0$$

$$c_s^2 = \frac{\gamma_e T_e + \gamma_i T_i}{m_i}$$

Lower hybrid waves

Assume $\mathbf{k} \| \mathbf{e}_x$, $k_z \to 0$ and $\Omega_i \ll \Omega_e$

$$\omega^2 = k_x^2 c_s^2 + |\Omega_i \Omega_e|$$

When $k_x \to 0$

$$\omega^2 = |\Omega_i \Omega_e| \equiv \omega_{LH}^2$$

Physical picture

- **Massive** ions move along $\mathbf{E}$
- **Light** electrons perform $\mathbf{E} \times \mathbf{B}$ along y and polarisation drift along $\mathbf{E}$
- Lower hybrid frequency is when electrons polarisation drift is equal to ion velocity along $\mathbf{E}$ to ensure quasi-neutrality.

Dawson, pp. 146-150

Limiting cases: propagation nearly perpendicular to B

After a few algebraic calculations…

$$1-\frac{\mathbf{k}^2c_s^2}{\omega^2}+\frac{\Omega_i}{\omega}\left\{-\left[\frac{\omega}{\Omega_i}-\frac{k_z^2}{k_x^2}\frac{\Omega_i}{\omega}\left(1-\frac{\omega^2}{\Omega_i^2}\right)\right]^{-1}+\left[\frac{\omega}{\Omega_e}-\frac{k_z^2}{k_x^2}\frac{\Omega_e}{\omega}\cancel{\left(1-\frac{\omega^2}{\Omega_e^2}\right)}\right]^{-1}\right\}=0$$

$$c_s^2=\frac{\gamma_eT_e+\gamma_iT_i}{m_i}$$

Simplified dispersion relation

Assume $k_x \gg k_z$, $\omega \simeq \Omega_i$ and $\Omega_i \ll \Omega_e$

$$1-\frac{\mathbf{k}^2c_s^2}{\omega^2}+\frac{\Omega_i}{\Omega_e}\frac{k_x^2}{k_z^2}-\left[\frac{\omega^2}{\Omega_i^2}-\frac{k_z^2}{k_x^2}\left(1-\frac{\omega^2}{\Omega_i^2}\right)\right]^{-1}=0$$

Electrostatic ion cyclotron waves

When $k_x/k_z \ll (m_i/m_e)^{1/2}$

$$\omega^2=kc_s^2+\Omega_i^2$$

Dawson, pp. 146-150